Quasi-static snap-through of a bistable beam

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1 Introduction

We consider a doubly clamped beam that has an initially curved shape analytically described by:

$$w_0 = \delta\phi_1(x) \quad \text{with} \quad \phi_1 = \frac{1}{2}\left(1 - \cos\frac{2\pi x}{L}\right) \tag{1}$$

with $\delta = 8\,\mu$m; $L = 600\,\mu$m is the distance between the the anchors; $t = 2\,\mu$m is the in-plane thickness. The out of plane thickness is $H = 22\,\mu$m. We assume linear elastic isotropic constitutive behaviour with Young modulus $E = 150000\,$MPa and volume density $\rho = 2330$Kg/m^3. The force P is exerted on the midspan of the beam.

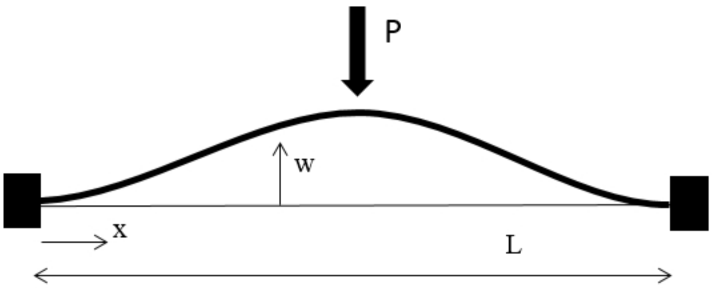

Figure 1: Doubly clamped beam. w is the beam deflection

An approximation of the symmetric deflections in statics of the beam can be obtained by solving the following form of the Principle of Virtual Power.

Find the deflection $w(x) \in \mathcal{C}'(0)$ such that, for $\forall\tilde{w} \in \mathcal{C}'(0)$:

$$\int_0^{L/2}\Big(EI(w'' - w_0'')\tilde{w}'' - N[w]w'\tilde{w}'\Big)\mathrm{d}x = -\frac{P}{2}\tilde{w}(L/2) \tag{2}$$

The space $\mathcal{C}'(0)$ is here the space of functions w with continuous first derivative and $w = w' = 0$ in $x = 0$ and $w' = 0$ in $x = L/2$. $A = Ht$ is the cross section area; $I = 1/12Ht^3$ is the inertia modulus; EI is the flexural stiffness. Moreover:

$$N[w] = \frac{EA}{2L}\int_0^L \left((w_0')^2 - (w')^2\right)\mathrm{d}x$$

is the compressive axial force assumed independent of the position along the beam.

2 Newton-Raphson procedure

Starting from $P = 0$ we increase the load by ΔP for each loading step. Let us focus on a specific loading step and set up a Newton-Raphson procedure for its solution.

1) If $w_{[k]}$ denotes the current estimate at iteration [k] in the loading step and δw the increment over the iteration, express the residuum and the tangent linear operator (at a continuum level, before discretization)

R1) The axial force N is non-linear. Its linearization gives:

$$
\begin{aligned}
N[w_{[k]} + \delta w] &= \frac{EA}{L}\int_0^{L/2} \Big((w_0')^2 - (w_{[k]}' + \delta w')^2\Big)\,\mathrm{d}x \\
&= N[w_{[k]}] - \frac{2EA}{L}\int_0^{L/2} w_{[k]}'\delta w'\mathrm{d}x + o(\delta w)
\end{aligned}
$$

Now, let us define the residuum:

$$
\mathcal{R}[w^{[k]};\tilde{w}] = \int_0^{L/2}\Big(EI(\delta w_{[k]}'' - w_0'')\tilde{w}'' - N[w_{[k]}]w_{[k]}'\tilde{w}'\Big)\mathrm{d}x + \frac{P}{2}\tilde{w}(L/2)
$$

The global linearization of eq.(2) yields:

$$
\int_0^{L/2}\Big(EI(\delta w''\tilde{w}'' - N[w_{[k]}]\delta w'\tilde{w}'\Big)\mathrm{d}x \\
+ \frac{2EA}{L}\int_0^{L/2} w_{[k]}'\delta w'\mathrm{d}x \int_0^{L/2} w_{[k]}'\tilde{w}'\mathrm{d}x = -\mathcal{R}[w_{[k]};\tilde{w}]
$$

The LHS is the tangent linear operator

3 Discretization

The half-beam is discretized as a segment partitioned into uniform elements of length h with Hermite cubic shape functions.

$$
w_h = N_1(a)W^{(1)} + N_2(a)h\Phi^{(1)} + N_3(a)W^{(2)} + N_4(a)h\Phi^{(2)} \tag{3}
$$

where a is the master space parameter $0 \leq a \leq 1$ and (see Figure 2):

$$
\begin{aligned}
N_1 &= 2a^3 - 3a^2 + 1 \\
N_2 &= (a^3 - 2a^2 + a) \\
N_3 &= -2a^3 + 3a^2 \\
N_4 &= (a^3 - a^2)
\end{aligned}
$$

Each element has two nodes and $W^{(i)}$ is the nodal deflection, $\Phi^{(i)}$ is the nodal slope. Also w_0 is discretized according to eq.(3) using the exact nodal values from eq.(1)

2) Is eq.(3) correct dimensionally? Explain

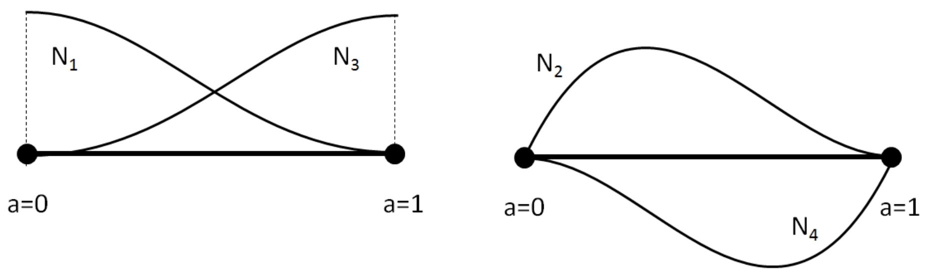

Figure 2: Hermite cubic shape functions

R2) Yes it is. Indeed shape functions are non-dimensional. W and $h\Phi$ have the dimensions of a length

On the contrary, for the geometry we simply use a linear mapping from the parameter space (the element is *not* isoparametric).
Suppose that two matrices are available: matrix $[M_D]$, such that:

$$[M_D]_{ij} = \int_0^1 N_i'(a) N_j'(a) \mathrm{d}a$$

and matrix $[M_{DD}]$, such that:

$$[M_{DD}]_{ij} = \int_0^1 N_i''(a) N_j''(a) \mathrm{d}a$$

3) *Express the contribution of a generic element to the residuum and to the tangent linear operator in terms of these matrices, nodal values and of geometrical and material parameters*

R3) On a given element define the list:

$$\{W_e\} = \{W^1,\, h\Phi^1,\, W^2,\, h\Phi^2\}$$

of the nodal degrees of freedom. Similary for $\{\delta W_e\}$ or $\{\tilde{W}_e\}$. As an example:

$$\int_0^h w'_{[k]} \delta w' \mathrm{d}x = \frac{1}{h} \{W_e^{[k]}\}[M_D]\{\delta W_e\}$$
$$\int_0^h w''_{[k]} \delta w'' \mathrm{d}x = \frac{1}{h^3} \{W_e^{[k]}\}[M_{DD}]\{\delta W_e\}$$

For this reason we redefine:

$$[\bar{M}_D] = \frac{1}{h}[M_D] \qquad [\bar{M}_{DD}] = \frac{1}{h^3}[M_{DD}]$$

A given element contributes to many terms. Let us start with the axial force contribution:

$$\begin{aligned} N^e[w_{[k]}] &= \frac{EA}{L} \int_0^h \left((w_0')^2 - (w'_{[k]})^2 \right) \mathrm{d}x \\ &= \frac{EA}{L} \left(\{W_e^0\}[\bar{M}_D]\{W_e^0\} - \{W_e^{[k]}\}[\bar{M}_D]\{W_e^{[k]}\} \right) \end{aligned}$$

and the total axial force is simply the sum of the elemental contributions.

The contribution of the element to the first terms of the tangent operator is:

$$\int_0^h \Big(EI(\delta w''\tilde{w}'' - N[w_{[k]}]\delta w'\tilde{w}'\Big)\mathrm{d}x$$
$$= EI\{\tilde{W}\}[\bar{M}_{DD}]\{\delta W_e\} - N[w_{[k]}]\{\tilde{W}_e\}[\bar{M}_D]\{\delta W_e\}$$

These terms must be assembled with a standard assembly procedure. The last term in the linear tangent operator is slightly different. It is the product of two almost identical terms. The first is:

$$\int_0^{L/2} w'_{[k]}\delta w'\mathrm{d}x = \sum_e \int_0^h w'_{[k]}\delta w'\mathrm{d}x$$

and:

$$\int_0^h w'_{[k]}\delta w'\mathrm{d}x = \{W_e^{[k]}\}[[\bar{M}_D]]\{\delta W_e\} = \{F_e\}^T\{\delta W_e\}$$

This will be assembled generating a global vector:

$$\int_0^{L/2} w'_{[k]}\delta w'\mathrm{d}x = \{F\}^T\{\delta W\}$$

Similary:

$$\int_0^{L/2} w'_{[k]}\tilde{w}'\mathrm{d}x = \{F\}^T\{\tilde{W}\}$$

The global assembly will then produce the term:

$$\int_0^{L/2} w'_{[k]}\delta w'\mathrm{d}x \int_0^{L/2} w'_{[k]}\tilde{w}'\mathrm{d}x = \{\tilde{W}\}\{F\}\{F\}^T\{\delta W\}$$

4 Numerical project

Upload file `NResame.m` from the personal page of Attilio Frangi (directory PhD FEM) in the the DICA web site (for fast access use Google search `frangi intranet dica`). In the file all the parameters are defined in a suitable unit systems. The skeleton of the code is also provided. This skeleton must be filled with missing parts as specified in the following points.
Read carefully the file through the definition of the matrices $[M_D]$ and $[M_{DD}]$ (the same matrices introduced in the previous section).

4) *Comment the definition of* `dof`. *Is it correct?*

R4) No, it is wrong. The last term must vanish since it corresponds to the boundary condition that the tangent vanishes.

Each NR iteration is divided in 4 parts. The computation of the current axial force, the computation of the tangent stiffness matrix, the computation of the right hand side, the solution for the increment and update.

5) *Fill the missing line for the elemental contribution of the axial load*

6) *Write the assemblage of the stiffness matrix and of the rhs. Solve for the increment and update the solution*

7) *Run the code and comment*

R5-7) See the Matlab file

5 Inclusion of a constraint

We change perspective and replace the load control implemented in the previous section with a *displacement control*. To this aim we take the discretized system developed in the previous sections and treat P as an unknown. We hence need and additional equation, represented by the displacement control $\Delta w(L/2) = \Delta s$, with Δs given increment (e.g. $\Delta s = 0.5$). $\Delta w(L/2)$ represents the increment of the mid-span deflection in one load step. It is worth stressing that in this case $\Delta w(L/2) = \Delta s$ is NOT treated as a boundary condition, but as an additional equation to be solved in conjunction with eq.(2).

8) Express the new residuum and the new linear tangent operator before discretization (for the NR iterative procedure)

9) Starting from the Matlab file developed in previous points, make a copy and adapt it so as to implement the displacement control.

R8,R9) Answer not developed