Snap-through of a bistable beam

February 23, 2015

1 Introduction

We consider a doubly clamped beam that has an initially curved shape analytically described by:

$$w_0 = \delta\phi_1(x) \quad \text{with} \quad \phi_1 = \frac{1}{2}\left(1 - \cos\frac{2\pi x}{L}\right) \tag{1}$$

with $\delta = 10\,\mu\text{m}$; $L = 600\,\mu\text{m}$ is the distance between the the anchors; $t = 2\,\mu\text{m}$ is the in-plane thickness The out of plane thickness is $H = 22\,\mu\text{m}$.
A rigid mass with $M = 10^{-8}$ Kg is attached to the anchors via two springs having total stiffness $K = 5.5\,\text{N/m}$. The mass is initially nearly in contact with the beam with zero contact force.

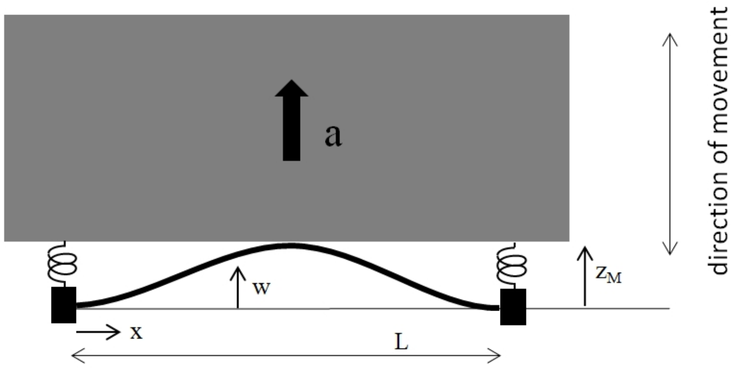

Figure 1: Rigid mass and doubly clamped beam. The coordinates system is attached to the anchors; z_M denotes the shuttle position; w the beam deflection

We work in frame rigidly attached to the anchors and assume that this frame is subjected to the external in-plane acceleration a aligned with the device as depicted in Figure 1:

$$a(t) = \frac{N_g}{2}(1 - \cos(2\pi f_0 t)) \quad t < \frac{1}{f_0} \qquad a(t) = 0 \quad t > \frac{1}{f_0}$$

with $f_0 = 1500\,\text{Hz}$. We also assume that the movement will occur along the same direction.
If contact develops between the mass and the beam (we assume pointwise contact in the middle of the beam), the mass will receive the force P. The force $-P$ will also be exerted by the mass on the midpoint of the beam.

Rigid mass. Globally, in a frame rigidly connected with the anchors, the shuttle dynamics is governed by the 1D model

$$M(\ddot{z}_M + a) + C\dot{z}_M + K(z_M - \delta) = P \tag{2}$$

where z_M denotes the mass coordinate and $C = 0.025\,\mu\text{N}\mu\text{s}/\mu\text{m}$ is a dissipation coefficient.

Slender beam. The dynamics of a generic slender beam under moderately large displacements and subjected to a distributed force $f(x)$ can be described in strong form by:

$$\rho A\ddot{w} + EI(w'''' - w_0'''') + N[w]w'' = f \qquad 0 < x < L \tag{3}$$

associated with suitable boundary conditions on w. $A = Ht$ is the cross section area; $I = 1/12Ht^3$ is the inertia modulus; ρA is the mass per unit length of the beam, EI is the flexural stiffness; and that rotational inertia of the cross sections has been neglected. Moreover:

$$N[w] = \frac{EA}{2L}\int_0^L \left((w_0')^2 - (w')^2\right) \mathrm{d}x$$

is the compressive axial force assumed independent of the position along the beam.

1) *Show that, for the doubly clamped beam of this exercise, the Principle of Virtual Power can be formulated as follows. Find the deflection $w(x) \in \mathcal{C}'(0)$ such that, for $\forall \tilde{w} \in \mathcal{C}'(0)$:*

$$\int_0^L \Big(\rho A(\ddot{w} + a)\tilde{w} + EI(w'' - w_0'')\tilde{w}'' - N[w]w'\tilde{w}'\Big)dx = -P\tilde{w}(L/2) \tag{4}$$

The space $\mathcal{C}'(0)$ is here the space of functions w with continuous first derivative and $w = w' = 0$ in $x = 0, L$.

It is worth stressing that the term $EI(w'' - w_0'')$ represents the bending moment in the beam. We assume linear elastic isotropic constitutive behaviour with Young modulus $E = 150000\,\text{MPa}$ and volume density $\rho = 2330\text{Kg/m}^3$.

Contact. The two equations (2) and (4) must be complemented with

$$z_M - w(L/2) \geq 0, \quad P > 0 \quad (z_M - w(L/2))P = 0 \tag{5}$$

governing "perfectly hard" contact.

2 Numerical project

In order to solve the set of coupled equations (2),(4),(5) we make a series of choices.

Penalty approach. A penalty approach is implemented to simulate contact. This consists in replacing perfectly hard contact with a contact force linearly increasing with interpenetration:

$$P = \lambda(w(L/2) - z_M)H(w(L/2) - z_M)$$

where H is the Heaviside function and λ is a positive large penalty coefficient (say $\lambda = 40\,\text{N/m}$).

Symmetry. Only symmetric deflections of the beam are considered. This can be obtained by analysing only half of the beam with suitable boundary conditions.

2) Specify the new boundary conditions and reformulate the overall problem

Discretization. The half-beam is discretized as a segment partitioned into 5 uniform elements with Hermite cubic shape functions. On a given element of length h:

$$w_h = N_1(a)W^{(1)} + N_2(a)\Phi^{(1)} + N_3(a)W^{(2)} + N_4(a)\Phi^{(2)}$$

where a is the master space parameter $0 \leq a \leq 1$ and (see Figure 2):

$$\begin{aligned} N_1 &= 2a^3 - 3a^2 + 1 \\ N_2 &= h(a^3 - 2a^2 + a) \\ N_3 &= -2a^3 + 3a^2 \\ N_4 &= h(a^3 - a^2) \end{aligned}$$

Each element has two nodes and $W^{(i)}$ is the nodal deflection, $\Phi^{(i)}$ is the nodal slope.
On the contrary for the geometry we simply use a linear mapping from the parameter space (the element is *not* isoparametric).

3) On a given physical element express x in terms of $x^{(1)}, h$ and a.

4) Compute w_h' and w_h'' on a given physical element

5) Which are the unknowns of the problem? How many? Under which conditions w_h and w_h' are elements of $\mathcal{C}_h'(0)$?

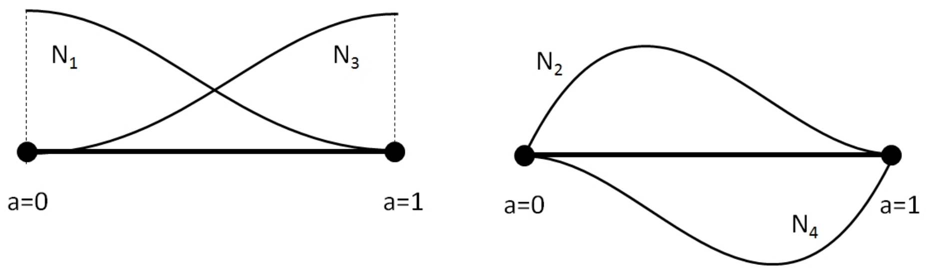

Figure 2: Hermite cubic shape functions

Integration scheme. The set of equations is integrated with an explicit central difference scheme.

6) Why does this choice represent a great simplification? Describe concisely (no more than 10 lines) the time step algorithm

Implementation.

7) *Develop a Matlab implementation. Analyse the case* $N_g = 1000g$*, where* g *is the acceleration of gravity and verify if the beam snaps to the second equilibrium configuration. Use the following units:* 10^{-6}*m for length;* 10^{-12}*Kg for mass;* 10^{-6}*s for time. In these units* $M = 10^4$, $K = 5.5$, $C = 0.025$, $\rho = 2300 \times 10^{-6}$, $E = 150000$, $f_0 = 1500 \times 10^{-6}$, $\lambda = 40$, $g = 9.81 \times 10^{-6}$