Simple shear flow of collisional granular-fluid mixtures

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Abstract

This work deals with the simple shear flow of neutrally buoyant, rigid, frictionless spheres immersed in a viscous fluid that exchange momentum through inelastic collisions. We show how kinetic theories are able to provide a full analytical description of the flow, once the influence of the viscous fluid is taken into account in a simple way through the dependence of the collisional coefficient of restitution on the Stokes number. This allows the capture of the characteristics of the experiments performed by Bagnold sixty years ago and the interpretation of the macro-viscous and inertial regimes described by the same author as the limits for the coefficient of restitution equal to zero and to the value valid in absence of the viscous fluid, respectively.

Introduction and theory

The simple shear flow (SSF) is the obvious configuration to study the response of fluids to deformation. Thus, it has been largely investigated, experimentally, numerically and theoretically, to determine the appropriate rheology of granular gases (Savage and Sayed 1984; Hanes and Inman 1985; GDR MiDi 2004; da Cruz et al. 2005; Mitarai and Nakanishi 2007; Orlando and Shen 2012).

Although theories that take into account the role of frictional contacts among deformable particles exist (Berzi et al. 2011), let us focus, for sake of simplicity, on the case of frictionless, rigid spheres. First, we briefly recall the case of dry SSF, already analyzed in great detail (GDR MiDi 2004; da Cruz et al. 2005; Mitarai and Nakanishi 2007).

A certain amount of granular material, characterized by a constant volume concentration ν and composed of mono-dispersed rigid spheres of diameter d and density ρ_p, is confined between two parallel plates, and homogeneously sheared ($\dot{\gamma}$ being the shear rate), in absence of external forces (Fig. 1). The shearing induces inter-particle collisions, that we characterize through a coefficient of restitution, e (ratio of pre- to post-collisional relative velocity between two colliding particles). Macroscopic shear stress, s, and pressure, p (the isotropic component of the normal stresses) result from the statistical average of the momentum exchange due to collisions (Goldhirsch 2003). Hence, s and p are unique functions of the five independent variables, ν, d, ρ_p, $\dot{\gamma}$ and e. Using the particle diameter and density and the shear rate to non-dimensionalize the problem, we reduce the number of independent variables to two, i.e., ν and e. Conversely, the shear stress and the pressure must be substituted by two non-dimensional numbers. The French group GDR MiDi (2004) have suggested to use the particle stress ratio, $\mu \equiv s/p$, and the inertial number, $I \equiv \dot{\gamma} d / \left[p / (\rho \nu) \right]^{1/2}$, respectively, for this purpose. The rheology of the dry granular material is fully determined once the two functions, $\mu = \mu(\nu, e)$ and $I = I(\nu, e)$, are known. That is, in dry condition, every value of concentration corresponds to a certain value of the particle stress ratio (Mitarai and Nakanishi 2007), given the value of the coefficient of restitution, which is a material property. Of course, one can use the inertial number as independent variable instead of the concentration, leading to the theoretically equivalent problem of determining the two functions $\mu = \mu(I, e)$ and $\nu = \nu(I, e)$. In experiments and numerical simulations, the two cases are distinguished and called concentration- and pressure-imposed SSF, respectively.

Let us see now what changes when the particles are immersed in a viscous fluid, as in Bagnold's pioneering experiments (Bagnold 1954). Bagnold's idea was to use neutrally buoyant spheres immersed in a fluid to eliminate the influence of gravity and approximate the ideal conditions of SSF. The presence of the interstitial fluid, though, introduces an additional variable to the problem, the fluid viscosity η. Hence, a non-dimensional number, representing the ratio of the particle inertia to the fluid viscous forces, must be included as an additional independent variable of the problem. This additional degree of freedom implies that, unlike the dry case, infinite values of the particle stress ratio are possible at a given concentration (Bagnold 1954).

The expression for μ can be easily obtained from kinetic theories (Mitarai and Nakanishi 2007), even when a viscous interstitial fluid is present. The pressure and the shear stress, in the dense limit (Jenkins and Berzi 2010), i.e., ν greater than say 0.4, and using the constitutive relations of Garzo and Dufty (1999), read

$$p = 2(1+e)\rho_p \nu^2 g_0 T, \tag{1}$$

and

$$s = \frac{8J}{5\pi^{1/2}}\rho_p \nu^2 g_0 T^{1/2} d\dot{\gamma}, \tag{2}$$

respectively, with

$$J = \frac{1+e}{2} + \frac{\pi}{4}\frac{(1+e)^2(3e-1)}{24-6(1-e)^2-5(1-e^2)}. \tag{3}$$

In Eqs. (1) and (2), T is the granular temperature, mean square of the particle velocity fluctuations, and g_0 is the radial distribution function at contact (Chapman and Cowling 1970). The balance of fluctuating energy of the particles provides the required equation to determine the granular temperature. In the case of SSF, that equation reduces to a balance between energy production and dissipation,

$$s\dot{\gamma} = \frac{12}{\pi^{1/2}}\left(1-e^2\right)\rho_p \nu^2 g_0 \frac{T^{3/2}}{L} + \Gamma_\eta \, , \tag{4}$$

where the first and the second term on the right hand side represents the rate of energy dissipation due to the inelastic collisions and the viscous drag on the particles (Hsu et al. 2004), respectively. There, *L* is the correlation length, the measure of the correlation among the particle velocity fluctuations whose effect is in diminishing the energy dissipated in collisions (Jenkins 2007). Its expression is, however, available only for some values of the coefficient of restitution (Jenkins and Berzi 2010, 2012); further investigations are needed to determine the complete dependence of *L* on *e*. In view of the above mentioned limitation, and for sake of simplicity, here we take $L = d$. An expression for Γ_η is, in principle, available (Hsu et al. 2004), although there are some issues concerning, for instance, the dependence of the drag on the particle concentration and the velocity fluctuations. Here, we prefer to adopt a simpler approach. The presence of the viscous fluid damps the inter-particle collisions, therefore enhancing the apparent inelasticity of contacts (Joseph et al 2001; Yang and Hunt 2006). As in Berzi (2011), we set $\Gamma_\eta = 0$ in Eq.(4), and take the coefficient of restitution dependent on the Stokes number, St, which represents the ratio of particle inertia to the viscous forces. With this, Eqs. (3) and (4) show that the granular temperature is an algebraic function of the shear rate,

$$T = \frac{2J}{15\left(1-e^2\right)} d^2 \dot{\gamma}^2 \, . \tag{5}$$

The particle stress ratio μ, therefore, results, from Eqs. (1), (2) and (5),

$$\mu = \left[\frac{24J\left(1-e^2\right)}{5\pi\left(1+e\right)^2}\right]^{1/2} . \tag{6}$$

The fact that μ results independent on ν in the dense limit, in contrast with experiments and numerical simulations on dry SSF (GDR MiDi 2004; Mitarai and Nakanishi 2007), is actually a strong argument in favor of the introduction of the correlation length in Eq. (4). We use the expression suggested by Barnocki and Davis (1988) for the dependence of the coefficient of restitution on the Stokes number,

$$\varepsilon - e \propto \frac{(1+\varepsilon)}{\mathrm{St}}, \tag{7}$$

where ε is the value of the coefficient of restitution in dry condition (i.e., when $\mathrm{St} \rightarrow \infty$), and $\mathrm{St} \propto \left(\rho_p d T^{1/2}\right)/\eta$, as in Berzi (2011). In the expression of the Stokes number, the square root of the granular temperature is taken to be a measure of the relative velocity between colliding particles (Chapman and Cowling 1970; Armanini et al. 2005). Bagnold (1954) used the following non-dimensional number,

$$\mathrm{N} = \left[\frac{1}{\left(\nu_M/\nu\right)^{1/3} - 1}\right]^{1/2} \frac{\rho_p d^2 \dot{\gamma}}{\eta} \tag{8}$$

to measure the importance of particle inertia to the fluid viscous forces, instead of the Stokes number. In Eq. (8), ν_M is the maximum packing concentration of the granular material (equal to 0.74 for rigid spheres). Using Eqs. (5), (7) and the expression for the Stokes number, Eq. (8) becomes

$$\mathrm{N} = a \left[\frac{1}{\left(\nu_M/\nu\right)^{1/3} - 1}\right]^{1/2} \frac{(1+\varepsilon)}{(\varepsilon - e)} \left[\frac{15\left(1-e^2\right)}{2J}\right]^{1/2}, \tag{9}$$

where we set the coefficient of proportionality, *a*, on the basis of comparisons with experiments.

Results and discussion

Eqs. (6) and (9) allow to determine the particle stress ratio, μ, and the Bagnold number, N, at a given concentration, for every value of the coefficient of restitution *e* from 0 to ε. Fig. 2 shows the experimental results of Bagnold (1954), as reported by Hunt et al. (2002), performed with neutrally buoyant wax spheres (ρ_p = 1000 kg/m^3 and d = 0.0013 m) in water (η = 0.001 Pa·s) or in a mixture of glycerin, water and alcohol (η = 0.007 Pa·s), in terms of μ versus N, for two different values of concentration (0.375 and 0.555).

We set ε = 0.95 - close to the values appropriate for spheres made of glass and cellulose acetate measured by Foerster et al. (1994) - and a = 5 to obtain the theoretical predictions of Fig. 2. We must emphasize that the exact quantitative agreement between the theory and the experiments is beyond the scope of the present work, so that the values of ε and a must be taken as purely indicative. Indeed, as already mentioned, the theory is not strictly rigorous, because of the rough simplification of neglecting the velocity correlation in Eq. (4), which is however present only at concentration larger than 0.49 (Jenkins 2007). On the other hand, also the correctness of Bagnold's experimental findings have been criticized (Hunt et al. 2002), thus making meaningless the construction, at this stage, of too refined a theory. Nonetheless, the present simple theory captures well the qualitative behavior of the experimental results. The decrease of the particle stress ratio with the Bagnold number can be explained with the associated increase of the apparent coefficient of restitution e; indeed, it is well known that lower values of the particle stress ratio pertain to less dissipative particles (Mitarai and Nakanishi 2007). Furthermore, it seems natural (Fig. 2) to relate the constant values of the particle stress ratio that Bagnold found appropriate for the macro-viscous and inertial regimes (Bagnold 1954), with the values obtained by setting $e = 0$ and $e = \varepsilon$ in Eq. (6), respectively. We must be cautious in making statements about the macro-viscous regime based on the present theory, given that we assume that the microscopic particle inertia still plays the major role in determining stresses (the constitutive expressions for the particle shear stress and pressure are indeed both proportional to the particle density). However, the particle stress ratio does not depend on the particle density, and Fig. 2 seems to suggest that kinetic theories may be used also to obtain information about the behavior of the particle stress ratio at vanishingly small values of the Stokes number (or, equivalently, of the Bagnold number). We leave the analysis to future works.

Conclusions

We have focused on the simple shear flow of granular-fluid mixtures to show that their behavior can be described in the framework of kinetic theories, if the influence of the fluid viscosity on the collisional coefficient of restitution is taken into account through its dependence on the Stokes number (the measure of the ratio between the particle inertia and the fluid viscous forces). Kinetic theories predict that the ratio of particle shear stress to particle pressure decreases when the coefficient of restitution increases; given that the latter monotonically increases with the Stokes number, this explains why the experimental particle stress ratio decreases with the physically-equivalent Bagnold number. The present analysis also suggests that the well known macro-viscous and inertial regimes introduced by Bagnold (1954), whose work is at the origin of the modern literature on both debris flows (Berzi et al. 2010) and dense suspensions (Boyer et al. 2011), can be interpreted as the limits for the Stokes-dependent coefficient of restitution that goes to zero and to the value of the dry granular material, respectively.

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Figure 1

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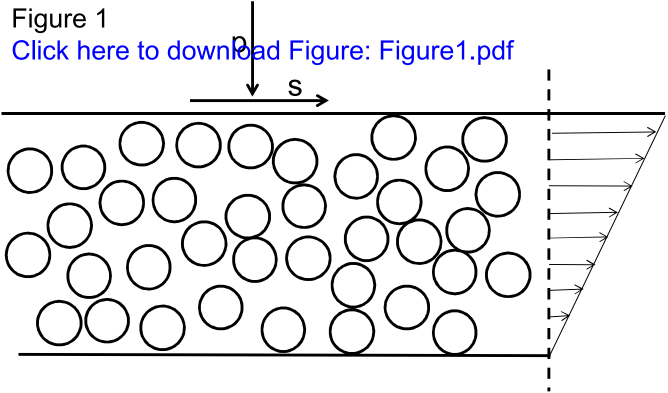

Figure 2

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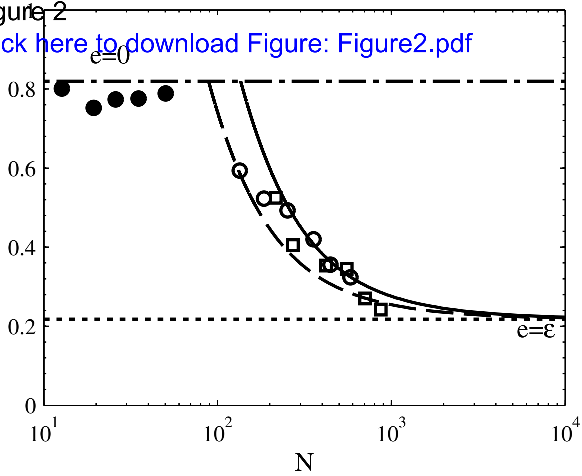

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Figure 1. Sketch of the SSF configuration.

Figure 2. Experimental (symbols, after Bagnold 1954) and theoretical (lines) particle stress ratio as function of the Bagnold number. Experiments refer to wax spheres in water (open symbols) and wax spheres in a mixture of water, glycerin and alcohol (filled symbols), for $\nu = 0.375$ (squares) and $\nu = 0.555$ (circles). The theoretical predictions are for $\nu = 0.40$ (dashed line) and $\nu = 0.56$ (solid line). Also shown are the predicted particle stress ratios when $e = 0$ (dot-dashed line) and $e = \varepsilon$ (dotted line).