Surface flows of inelastic spheres

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We study flows of inelastic spheres on the surface of an erodible bed between frictional sidewalls and distinguish two regions in such flows: a dilute, diffuse region, neighboring the free surface, for which we solve a boundary-value problem based on the kinetic theory, and a dense algebraic layer, in which there is an approximate algebraic balance between production and dissipation of fluctuation energy. We take into account correlated motions between the particles at high volume fractions and employ the trapezoidal rule to solve, in an approximate way, for the flow quantities in the diffuse layer. Using boundary conditions of no-slip and yield at the bed and vanishing of the stresses and the energy flux at the free surface, we obtain analytical predictions of flow depth and mass flow rate that compare favorably with the results of experiments performed on glass spheres flowing on the surface of a heap and in half-filled rotating drums. © *2011 American Institute of Physics*. [doi:10.1063/1.3532838]

I. INTRODUCTION

In the recent years, surface flows of dry granular material over an apparently static bed have received great attention. Indeed, because the particle volume fraction varies through the flow field, this relatively simple flow configuration permits the simultaneous observation of several different regimes. Near the free surface, there is a dilute region, in which the particle interactions are essentially chaotic, binary, instantaneous collisions;[1,2] in moving from this region into the interior, there is a transition to a dense collisional regime, in which correlated motion seems to play a fundamental role;[3–6] and, finally, at the bed, a shear rigidity develops[7] and particles of the bed interact with jostling, enduring contacts that permit creep with an exponentially decaying velocity profile.[8]

So far, two different experimental configurations have been employed to study surface granular flows. In the first one, particles are continuously fed to the top of a heap with constant mass flow rate;[9,10] in the other one, a partially filled drum is rotated at a constant angular velocity around a horizontal axis.[11–13] In the former, a steady, fully developed flow is obtained, while in the latter, the flow is steady, but not fully developed. Nevertheless, experimental results in the two devices agree, and this indicates that unlike flow over rigid beds,[14,15] the angle of inclination of the free surface and the depth of the flow above the bed are completely determined by the mass flow rate over the heap and the angular velocity of the rotating drum.

Taberlet *et al.*[9] show that the presence of frictional sidewalls plays a fundamental role in controlling surface granular flows, permitting flows at angles of inclination of the free surface much higher than the angle of repose of the granular material. Jop *et al.*[10] perform experiments on heap flows by changing, among other parameters, the distance between the sidewalls; they suggest a scaling that employs this distance to obtain universal relations between flow depth, average velocity, and angle of inclination of the free surface. They also derive analytical expressions for these relations using a phenomenological local rheology, valid for dense flows, introduced by the French group GDR MiDi.[16] This rheology links the ratio of shear to normal stress to the so-called inertial parameter (the ratio of time scales associated with particle motions perpendicular and parallel to the flow, respectively). Félix *et al.*[13] perform experiments in rotating drums by changing the diameter and the width of the drum and the particle diameter; they conclude that the results are sensitive to the ratio between the drum and particle diameters and independent of the drum width, and that it is not possible to deduce a local particle rheology from experiments in a rotating drum.

Recently, Jenkins and Berzi[17] extended an existing kinetic theory for identical, nearly elastic, frictionless spheres to identical, very dissipative, frictional spheres, taking into account the possibility of correlated motion between the particles. Assuming that an algebraic balance between production and dissipation of fluctuation energy holds everywhere and that the flow regime is dense, they were able to derive analytical solutions for dense flows between frictional sidewalls that compared well with measurements in the experiments of Jop *et al.*[10] It has been shown[18,19] that such an algebraic balance applies in a core region, roughly five to ten diameters away from the top and bottom boundaries.

Here, we extend the work of Jenkins and Berzi[17] to take into account the possible presence of a diffuse region close to the free surface. In this region, the divergence of the flux of fluctuation energy in the energy balance for the particles is not negligible; so we solve the differential equations of the kinetic theory in an approximate way, using the trapezoidal rule of integration and boundary conditions at the free surface and the top of the dense region. This permits us to obtain an analytical solution for the diffuse region that we combine with the analytical solution for the dense, core re-

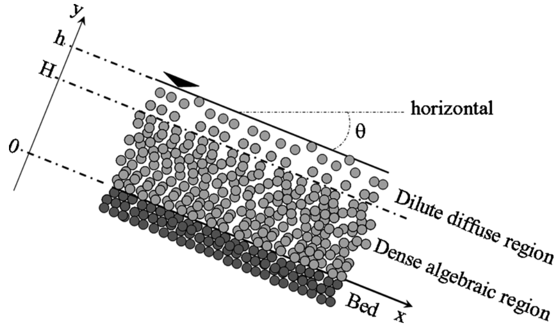

FIG. 1. Sketch of the flow configuration with the frame of reference.

gion already obtained in Jenkins and Berzi.[17] We indicate the influence of the model parameters on these, show that the predictions of the theory are in excellent agreement with the experiments of Félix *et al.*,[13] and give an explanation for the apparent dependence of their results on the geometry of the drum. Finally, we use the trapezoidal rule to provide a physical explanation for the observed dependence of the limiting inclination angle for steady flow on the distance between the sidewalls.

II. THEORY

A sketch of the flow configuration is given in Fig. 1. Here, and in what follows, all quantities are made dimensionless using the particle diameter, the mass density of the particle material, and the gravitational acceleration. The flow is assumed to be steady and fully developed in the x-direction and inclined of an angle θ with respect to the horizontal; the y coordinate is taken to be perpendicular to the free surface, with $y=0$ at the bed and $y=h$ at the free surface. There is a dilute, diffuse region, for $H \leq y \leq h$, and a dense core, of extent H in which there is an algebraic balance between production and dissipation of fluctuation energy. In what follows, we neglect all terms involving spatial variation along the spanwise direction. As we will see, this permits an analytical solution to the flow. Some indication of the validity of this approximation is given by the velocity profiles at the free surface reported in Ref. 10, although its final assessment relies essentially on comparison with experiments.

The balances of momentum parallel and perpendicular to the flow are

$$\frac{\mathrm{d}s}{\mathrm{d}y} = -\nu \sin\theta + 2\mu_w \frac{p}{W} \qquad (1)$$

and

$$\frac{\mathrm{d}p}{\mathrm{d}y} = -\nu \cos\theta, \qquad (2)$$

where s is the particle shear stress, p is the particle pressure, and ν is the particle volume fraction. In Eq. (1), we have incorporated frictional sliding at the walls, with friction coefficient μ_w, in an approximate way by averaging through the distance W between the sidewalls, as do Taberlet *et al.*[9] and Jop *et al.*[10] We set the value of μ_w on the basis of comparisons with experiments, and note that it may be influenced by the slight anisotropy in the normal stresses seen in numerical simulations.[20]

TABLE I. Summary of the constitutive relations.

$$p = 4\nu G F T$$

$$G \equiv \begin{cases} \nu(1-\nu/2)/(1-\nu)^3 & \text{if } \nu \leq 0.49 \\ 0.63\nu/(0.60-\nu) & \text{if } 0.49 \leq \nu < 0.60 \end{cases}$$

$$F = (1+e)/2 + 1/(4G)$$

$$\frac{1-e^2}{4} \equiv \frac{1-e_n{}^2}{4} + \frac{1+\beta_0}{7} - \left(\frac{1+\beta_0}{7}\right)^2 \times \left[1 + \frac{5(1+\beta_0)}{14-5(1+\beta_0)}\right]$$

$$s = \mu \frac{\mathrm{d}u}{\mathrm{d}y}$$

$$\mu = (2J/5\pi^{1/2})p/(FT^{1/2})$$

$$J = \frac{(1+e)}{2} + \frac{\pi}{32}\frac{[5+2(1+e)(3e-1)G][5+4(1+e)G]}{[24-6(1-e)^2-5(1-e^2)]G^2}$$

$$\Gamma = \frac{12}{\pi^{1/2}}\frac{\nu G}{L}(1-e^2)T^{3/2}$$

$$L = \max\left[1, \left(\frac{1}{2}c\frac{G^{1/3}}{T^{1/2}}\frac{\mathrm{d}u}{\mathrm{d}y}\right)\right]$$

The balance of fluctuation energy is

$$\frac{\mathrm{d}Q}{\mathrm{d}y} = s\frac{\mathrm{d}u}{\mathrm{d}y} - \Gamma, \qquad (3)$$

where Q is the energy flux in the y-direction, u is the particle velocity in the x-direction, and Γ is the rate of collisional dissipation. When the term on the left hand side of Eq. (3) is negligible, the production of fluctuation energy due to the working of the stress equals the dissipation due to collisions; we refer to this as the algebraic balance.

To close the problem, we employ the constitutive relations derived from the kinetic theory of Garzó and Dufty[21] for frictionless, inelastic spheres, neglecting the small terms introduced by their function c^*, as do Jenkins and Berzi.[17] The constitutive relations are provided in Table I. There, T is the granular temperature (one-third the mean square of the velocity fluctuations), G is the product of ν and the radial distribution function at contact, e_n is the coefficient of normal restitution, and β_0 is the coefficient of tangential restitution in a sticking collision. As in Ref. 17, we take e to be an effective coefficient of restitution that incorporates the effects of friction in the limit of infinite friction.[22] For $\nu \leq 0.49$, we use the radial distribution function determined by Carnahan and Starling[23] in numerical simulations at moderate volume fractions; for $0.49 \leq \nu < 0.60$, we adopt, instead, a modified version of the Torquato's radial distribution function,[24] with a singularity at a volume fraction less than 0.64. As shown by Mitarai and Nakanishi[5] and Kumaran,[6] the singularity in the radial distribution function for particles in a steady, homogeneous shearing flow depends on the restitution coefficient. We assume here that the singularity is at a volume fraction of 0.60, as do Jenkins and Berzi.[17] This seems appropriate for $e=0.7$,[5] and is very close to the values between 0.58 and 0.59 found by Kumaran.[6] However, as we will see, the value of G is obtained from its relation to the ratio of shear and normal stress; hence, the particular form of the radial distribution function adopted influences only the evaluation of the volume fraction, which we assume, for simplicity, to be zero at the top of the dilute, diffuse region and constant in the dense region.

Correlated motion between particles that are likely to occur when the flow is dense is taken into account through a correlation length L that appears in the constitutive expression of the rate of collisional dissipation.[3,4] In the constitutive expression of L reported in Table I, c is a constant of order unity. The argument of Jenkins[3,4] was that in dense shearing flows, the particles are forced into overlapping or chattering contact along the principal compressive axis; this clustering would explain the overestimation of the rate of collisional dissipation by the classical kinetic theories observed in numerical simulations on dense shear flow of disks.[18] It accounts, in a phenomenological way, for the overestimation of the usual measure, T, of the strength of the velocity fluctuations observed in numerical simulations of hard spheres in dense flows,[5] shown by Kumaran[6] to be related to the breakdown in the assumption of molecular chaos and the departure of distribution of the fluctuations in the relative velocity of colliding particles from the form associated with this assumption.

III. APPROXIMATE ANALYTICAL SOLUTION

As already mentioned, in order to obtain an approximate analytical solution for the flow of Fig. 1, we treat the dense, algebraic region and the dilute, diffuse region separately. For the latter, we integrate the three balance equations (1)–(3) and the constitutive relations for the shear stress and the energy flux of Table I using the trapezoidal rule: $\int_a^b \psi dy \approx (b-a)(\psi_a+\psi_b)/2$. From this point onward, the subscript indicates the position y at which the generic variable ψ is evaluated. The decomposition of the flow domain into separated regions was previously made by Louge[25] and Kumaran[26] when treating dense, inclined, granular flows on rigid beds in the absence of sidewalls. In addition to considering flows over erodible beds and including lateral confinement, the present work differs from these in the use of the trapezoidal rule.

The trapezoidal rule approximates profiles as varying linearly between their end points; consequently, the associated error is proportional to the curvature in the profile. Its use permits us to obtain simple analytical solutions to algebraic equations, and it does not require that the diffuse layer be a boundary layer[26] or to be dense.[25] Also, in contrast to Louge[25] and Kumaran,[26] we employ constitutive relations of a kinetic theory with estimates of the coefficients obtained from comparison of predictions and the experiments on inclined flows over rigid, bumpy boundaries in the absence of sidewalls.[14]

A. Dilute, diffuse region

We use the vanishing of the stresses and the energy flux at the free surface as boundary conditions. From Eqs. (1) and (2) and the boundary condition $p_h=0$, we obtain

$$\frac{s}{p} = \tan\theta - \mu_w \frac{p}{\nu W \cos\theta}. \tag{4}$$

This relation holds everywhere in the flow and not only in the dilute, diffuse region.

We now assume that the volume fraction at the free surface is very small and that the algebraic layer is entirely dense with $L=1$ at $y=H$. When we neglect the divergence of Q and employ the constitutive relation for L in the dense algebraic layer, the energy balance (3) provides an algebraic relation between granular temperature and shear rate,

$$\left(\frac{u'}{T^{1/2}}\right)^3 = \frac{15}{J}\frac{(1-e^2)}{cG^{1/3}}. \tag{5}$$

We use this to express L and s/p as functions of ν and e (see Jenkins and Berzi[17] for more details) as

$$L = \frac{1}{2}\left[\frac{15}{J}(1-e^2)c^2\right]^{1/3} G^{2/9} \tag{6}$$

and

$$\frac{s}{p} = \frac{4J}{5\pi^{1/2}}\frac{1}{(1+e)}\left[\frac{15}{J}\frac{(1-e^2)}{c}\right]^{1/3}\frac{1}{G^{1/9}}. \tag{7}$$

Employing Eq. (6) in Eq. (7), with the condition $L_H=1$, we obtain the value k of the stress ratio s/p at $y=H$,

$$k = \left(\frac{24 J_H}{5\pi}\frac{1-e}{1+e}\right)^{1/2}, \tag{8}$$

where, upon taking the dense limit in the expression for J of Table I, $J_H=(1+e)/2+(\pi/4)(3e-1)(1+e)^2/[24-(1-e)\times(11-e)]$. The value of k is, therefore, solely determined by the value of the coefficient of restitution.

If we employ the trapezoidal rule in Eq. (1) with $p_h=0$, $\nu_h=0$, and taking ν_H to be 0.6,

$$p_H = 0.3(h-H)\cos\theta. \tag{9}$$

Then, Eqs. (8) and (9) used in Eq. (4) give us the value of the depth of the dilute, diffuse region,

$$h - H = 2\frac{\max[k,\tan\theta]-k}{\mu_w}W, \tag{10}$$

in which we have accounted for the fact that a dilute, diffuse region is present only if $\tan\theta \geq k$.

Finally, from the definition of volume flow rate per unit width, $q_y=\int_0^y u\nu dy$, we obtain

$$q_h = q_H + 0.3u_H(h-H), \tag{11}$$

where the values of q_H and u_H are determined as part of the solution for the dense, algebraic region.

B. Dense, algebraic region

Given that the flow is dense in the algebraic region, the pressure distribution is given to a good approximation by

$$p = p_H + 0.6(H - y)\cos\theta. \quad (12)$$

As do Jenkins and Berzi,[17] we assume that the collisional kinetic theory that we employ ceases to be valid at volume fractions at which rate-independent contributions to the stress begin to develop. Here, we assume that a rate-independent shear rigidity characterizes the bed and take α to be a characteristic ratio of shear to normal stress there. It is a parameter of the model that can be interpreted as the angle of repose of the granular material; it coincides with the limiting, or stopping, angle for steady flow in a channel of infinite width. At the end of the section, we indicate how the presence of the sidewalls causes this stopping angle to increase and, thus, to differ from the angle of repose.

Using Eqs. (9), (10), and (12) in Eq. (4) gives, for $y=0$,

$$H = \frac{\min[k,\tan\theta] - \alpha}{\mu_w} W. \quad (13)$$

Equation (13) provides the depth of the dense algebraic region, once the values of the stress ratio at the extremes are known, with the minimum function taking into account the possibility that no dilute, diffuse region is present.

From the constitutive relations for the shear stress and for the pressure of Table I, we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \left[\frac{25\pi F}{16J^2\nu G}\right]^{1/2}\frac{s}{p}p^{1/2}. \quad (14)$$

Following Jenkins and Berzi,[17] in Eq. (14) we replace the function G of s/p in Eq. (7) by its value $\bar{G}$ at the average value $\overline{s/p}$ of s/p, where

$$\overline{\frac{s}{p}} = \min[k,\tan\theta] - \frac{\mu_w}{2}\frac{H-\delta}{W}. \quad (15)$$

Then, upon employing Eqs. (2) and (4) in Eq. (14) and taking the dense limit for the coefficients F and J with $\nu=0.6$,

$$\frac{\mathrm{d}u}{\mathrm{d}p} = -A\left(\tan\theta - \mu_w\frac{p}{0.6W\cos\theta}\right)p^{1/2}, \quad (16)$$

where $A=[25\pi(1+e)]^{1/2}/(32J_H^2 0.6^3\bar{G}\cos^2\theta)^{1/2}$. Integrating Eq. (16), with the boundary condition $u=0$, when $p=p_0$, with Eq. (12) providing p_0, gives

$$u = \frac{2}{3}A\tan\theta({p_0}^{3/2} - p^{3/2}) - \frac{2}{5}\frac{\mu_w A}{0.6W\cos\theta}({p_0}^{5/2} - p^{5/2}). \quad (17)$$

Then, with $p=p_H$, we can obtain u_H in Eq. (11) from Eq. (17).

The integration of Eq. (17) between $y=0$ and $y=H$ provides the value of the flow rate q_H,

$$q_H = \frac{2A{p_0}^{5/2}\tan\theta}{35\cos\theta}\left\{\frac{7}{3}\left[3 - 5\frac{p_H}{p_0} + 2\left(\frac{p_H}{p_0}\right)^{5/2}\right] - \frac{\mu_w p_0}{0.6W\sin\theta}\left[5 - 7\frac{p_H}{p_0} + 2\left(\frac{p_H}{p_0}\right)^{7/2}\right]\right\}. \quad (18)$$

In summary, we assume we know the properties of the granular material: the restitution coefficient e, yield stress ratio α, and coefficient c of the correlation length, and the characteristics of the sidewalls: the gap W and wall coefficient μ_w, and first set the value of $\tan\theta$ at a value greater than the minimum provided by the analysis reported in Sec. III D. Then, we calculate from Eqs. (8), (10), and (13) the values of k, h-H, and H, respectively; with these, the depth of the flow layer h can be determined as a unique function of the inclination of the free surface, a distinctive feature of granular flows over erodible beds. Then, the pressures p_H and p_0 are obtained from Eqs. (9) and (12); with these, u_H and q_H from Eqs. (17) and (18), and, finally, q_h from Eq. (11). In this way, we obtain the global quantities $\tan\theta$, h, and $q=q_h$ to be compared with existing experiments on flows of glass spheres at the surface of a heap and in rotating drums.[10,17]

C. Results

We first study the sensitivity of the approximate analytical solution to the parameters (Fig. 2). We use, as a reference set of parameters, $e=0.60$, $\alpha=0.40$, $c=0.50$, and $\mu_w=0.25$, as previously adopted by Jenkins and Berzi[17] for flows of glass spheres between glass sidewalls. With these, $k=0.58$. For $\tan\theta<k$, only the dense, algebraic region is present, and its depth increases linearly with $\tan\theta$. For $\tan\theta\geq k$, the dense, algebraic region has a constant depth and a dilute, diffuse region develops, with a depth that increases linearly with $\tan\theta$. The analytical solution, giving the flow rate as a function of the inclination angle, depicted in Fig. 2 refers to surface flows in a channel of width $W=57$, for which experimental results are available.[10] The solution depends on the friction coefficient of the sidewalls and the value of the stress ratio at the bed [Figs. 2(c) and 2(d)]. As already mentioned, the latter has a clear physical meaning as the angle of repose of the granular material in a wide channel. Consequently, it can easily be inferred from the experiments as the limiting value of the angle of inclination as q tends to zero. Jop *et al.*[10] suggested a value of 0.18 for μ_w, based on tilting board experiments; however, the slightly higher value adopted here, and in Ref. 17, permits a better fit to the relation between flow depth and flow rate measured in the experiments.

Jop *et al.*[10] performed experiments with glass spheres flowing over a heap, varying the distance W, between the vertical sidewalls made of glass. Assuming, as we do, that the sidewalls transmit a frictional force to the flow and that the GDR MiDi rheology holds,[16] they showed that the flow depth, the velocity, and flow rate per unit width must scale with W, $W^{3/2}$, and $W^{5/2}$, respectively. In the GDR MiDi rheology, the stress ratio is taken to be a function, to be determined through comparisons with experiments, of the inertial parameter—the ratio between time scales associated with motion parallel and normal to the flow, respectively. It can be easily shown[4,17] that in the context of the phenomenological extension to the kinetic theory, the GDR MiDi rheology is a direct consequence of the algebraic balance between production and dissipation in the balance of fluctuation energy. Hence, more generally, we can say that the scaling suggested by Jop *et al.*[10] holds when the algebraic layer occupies most of the flow.

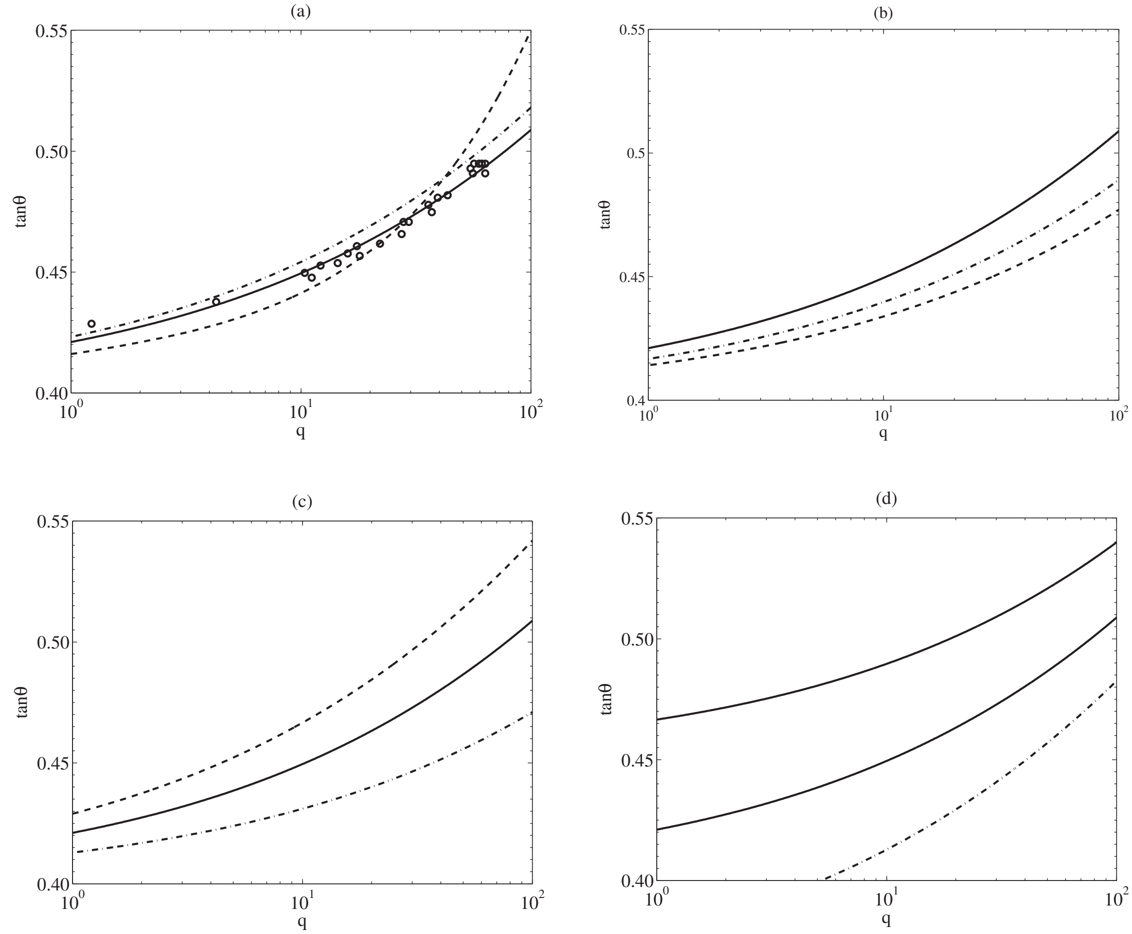

FIG. 2. Influence on the approximate analytical solution for flow rate vs the angle of inclination for a channel of $W=57$ of (a) the coefficient of restitution ($e=0.4$, dot-dashed line; $e=0.8$, dashed line). (b) The parameter c in the correlation length ($c=0.75$, dot-dashed line; $c=1.00$, dashed line). (c) The friction coefficients of the walls ($\mu_w=0.15$, dot-dashed line; $\mu_w=0.35$, dashed line). (d) The stress ratio at the bed ($\alpha=0.35$, dot-dashed line; $\alpha=0.45$, dashed line). In all the plots, the solid line represents the analytical solution obtained using the reference set of parameters. In (a), the experimental results (circles) of Jop *et al.* (Ref. 10) are also shown.

Our assumption of negligible particle volume fraction at the free surface has important consequences on the scaling when a dilute, diffuse region is present, that is, when $\tan\theta \geq k$. Then, the flow rate in the dilute, diffuse region [Eq. (11)] is determined from quantities in the dense, algebraic region, in which the scaling of Jop *et al.*[10] applies. Given that the depth of the dilute, diffuse region [Eq. (10)] also scales with W, we conclude that the scaling of Jop *et al.*[10] should also hold at the highest values of the angle of inclination when the dilute, diffuse region is present. To check this, we make comparisons with the experiments performed by Félix *et al.*[13] on rotating drums half-filled with glass spheres, using the reference set of parameters to obtain the approximate analytical solution. In the experiments, Félix *et al.*[13] used annuli of diameter D and width W, ranging from 47 to 7400 and 10 to 100, respectively. For a given angular velocity Ω, they measured the depth h of the flowing layer as the distance between the free surface and the lowest static point in the laboratory frame of reference. The associated flow rate per unit width is

$$q = 0.60\Omega\frac{D^2}{8}. \quad (19)$$

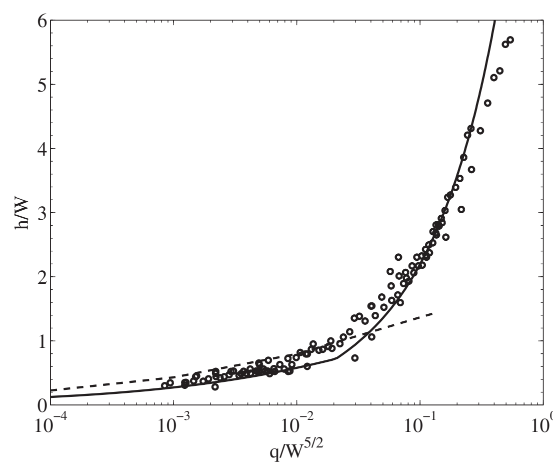

FIG. 3. Scaled flow depth against scaled flow rate per unit width: the experiment (circles) of Félix *et al.* (Ref. 13), the present theoretical treatment (solid line), and the analytical solution (dashed line) of Jop *et al.* (Ref. 10).

In Fig. 3 we show the theoretical and experimental flow depth h/W against the scaled flow rate per unit width, $q/W^{5/2}$. First, the scaling suggested by Jop *et al.*[10] holds, as anticipated, for $h/W \geq (k-\alpha)/\mu_w \approx 0.72$, corresponding to angles of inclination with tan $\theta \geq k$, for which a dilute, diffuse region is present. Second, the present theory is capable of predicting the experimental results in a striking way, although the definition of flow depth employed here, based on the value of the stress ratio for which shear rigidity occurs, differs from that adopted by Félix *et al.*[13] Also shown in Fig. 3 is the theoretical curve derived by Jop *et al.*[10] using the GDR MiDi rheology for the glass sphere parameters suggested by Forterre and Pouliquen.[27] As already mentioned, an approach based on such a local rheology fails in the presence of a dilute, diffuse region. Finally, the dependence on the drum diameter D of the experiments of Félix *et al.*[13] can be explained; indeed, by increasing D, Félix *et al.*[13] increased q, as shown in Eq. (19), and, therefore, generated flows in which the dilute, diffuse region became progressively more important.

D. Stopping angle

We show here how the trapezoidal rule can also be employed to derive an approximate analytical expression for the dependence of the stopping angle on the channel width, experimentally demonstrated by Grasselli and Herrmann[28] and Zhou *et al.*[29]

At the limiting inclinations associated with stopping, the boundary influences the dense region throughout its depth and there is no dilute, diffuse region. In this situation, we use the trapezoidal rule and the constitutive relation for the shear stress in the energy balance [Eq. (3)] to obtain

$$Q_h - Q_0 = (s_h{}^2/\mu_h - \Gamma_h + s_0{}^2/\mu_0 - \Gamma_0)h/2. \tag{20}$$

This should apply in the limit of small h when the distribution of the energy flux in the flow is approximately linear. Both the first and the second terms between brackets on the right hand side of Eq. (20) are proportional to $\nu_h{}^{1/2}p_h{}^{3/2}$, as it is easy to show by taking the dilute limit in the expressions of Table I: $J_h = 25\pi/32[24-6(1-e)^2-5(1-e^2)]^{-1}\nu_h^{-2}$, $G_h=\nu_h$, and $F_h=(4\nu_h)^{-1}$. At the interface with the bed, we employ the boundary condition for the energy flux derived by Jenkins and Askari,[30] slightly modified to include the correlation length at the bed,

$$Q_0 = -2p_0\frac{T_0^{1/2}}{L_0}\left[\frac{3(1-e)M_0}{\pi}\right]^{1/2}, \tag{21}$$

where, in the dense limit, $M_0=(1+e)/2+9\pi(1+e)^2(2e-1)/\{8[16-7(1-e)]\}$, and p_0 is evaluated from Eq. (12).

Using Eq. (21) in Eq. (20), and the expressions of Table I in the dense limit, with the boundary conditions $p_h=Q_h=0$ at the free surface, we obtain

$$h = \frac{16[3(1-e)M_0]^{1/2}}{5\pi(1+e)\alpha^2 L_0/J_H - 24(1-e)}. \tag{22}$$

We now assume, as does Jenkins,[3] that the flow stops when the depth h is equal to L_0; hence, the stopping height of the flow, i.e., the limiting value h_{stop} of the depth for the energy balance [Eq. (20)] to hold, results from Eq. (22),

$$h_{\text{stop}} = \frac{12(1-e)+\{144(1-e)^2+80\pi(1+e)\alpha^2[3(1-e)M_0]^{1/2}/J_H\}^{1/2}}{5\pi(1+e)\alpha^2/J_H}, \tag{23}$$

which is only a function of e and α. With this, from Eqs. (10) and (13), we obtain the stopping inclination angle for a surface granular flow,

$$\tan\theta_{\text{stop}} = \alpha + \frac{\mu_w}{W}h_{\text{stop}}. \tag{24}$$

The stopping angle is, therefore, a decreasing function of the distance between the sidewalls, as reported by Grasselli and Herrmann[28] and Zhou *et al.*[29] Furthermore, the dependence [Eq. (24)] of the stopping angle on W indicates that the scaling suggested by Jop *et al.*[10] should break down as tan θ approaches this value. Indeed, their experiments (e.g., their Fig. 11) confirm this observation.

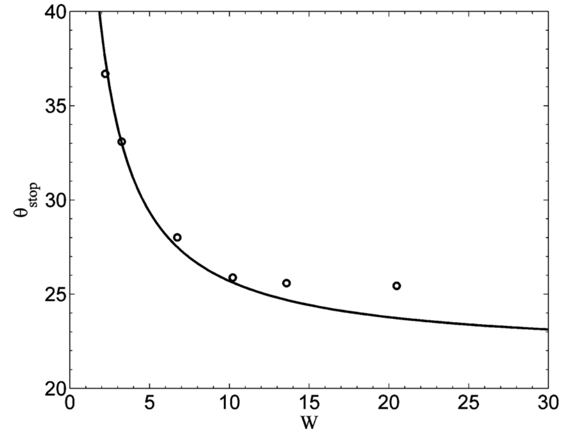

FIG. 4. Experimental [circles, from du Pont *et al.* (Ref. 31)] and theoretical (solid line) stopping angle vs distance between sidewalls.

In Fig. 4, we test the prediction [Eq. (24)] for the stopping angle of the particles as a function of W against the experiments[31] on 3 mm diameter glass spheres totally submerged in water. The agreement is very good, even though the value of α that we employ, which, once again, corresponds to the stopping angle when W is infinite, is smaller than that measured. However, as suggested by du Pont *et al.*,[31] this asymptotic value of the stopping angle depends on the way of constructing the heap and, therefore, the presence of water can play a role.

IV. CONCLUSION

We have studied steady flows of inelastic spheres at the surface of an erodible bed between frictional sidewalls. We have distinguished two regions in the flow: a dilute, diffuse region close to the free surface, in which we have solved the complete system of equations provided by the kinetic theory, and a dense, algebraic region below it, where the algebraic balance between production and dissipation of fluctuation energy holds. We have taken into account that correlated motion between the particles develops at high volume fraction,[3,4,17] and we have employed a simple integration rule to solve, in an approximate way, for the flow in the diffuse layer. Using appropriate boundary conditions of no-slip and yield at the bed and the vanishing of the stresses and the energy flux at the free surface, we have obtained an analytical description of the flow that compares favorably with the experimental results performed on glass spheres flowing at the surface of a heap[10] and in half-filled rotating drums.[13] We have also used the simple integration rule to derive an expression for the stopping angle of the granular material as a function of the distance between the sidewalls that agrees with existing experiments.[31]

The main result of the present theoretical treatment is (i) a local rheology for the particles can be employed only when the flow is entirely dense—that is, when no dilute, diffuse region is present; however, (ii) the scaling that makes use of the distance between the sidewalls also holds when a dilute, diffuse region is present, as in the experiments;[31] and (iii) the scaling breaks down as the flow approaches the stopping angle of the granular material, in accordance with the experiments.[10]

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